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A VARIATIONAL PRINCIPLE FOR CONSTRUCTING THE EQUATIONS OF ELASTOPLASTICITY FOR FINITE DEFORMATIONS†

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A variational principle of maximum dissipation of mechanical energy is proposed for constructing the governing equations of elastoplastic flow for finite deformations, based on the assumption that part of the dissipation is due to a change in the tensor of the internal variables. The required equations are obtained for the isothermal process by using a previously proposed subdivision of the complete metric tensor into elastic and plastic parts (but without invoking the idea of the rate of plastic deformation). The system of differential equations includes the equations for the tensor of the internal variables.

The governing equations for elastoplastic media with finite deformations were obtained in [1] which ensured that the subdivision of the total deformations into elastic and plastic parts and the stress tensor were independent of the deformation path in the elastic region. The idea of a rate of plastic deformation and of an objective derivative of the tensor with respect to time was not postulated. The Mises principle was used to close the system of equations for an active load, and a definition of the rate of plastic deformation was introduced which it was proposed to use to record the kinematic equations of the parameters of the history. However, for finite deformations there is not a sufficient basis for using the Mises principle, in particular, in view of the existence of different possibilities of introducing the idea of the rate of plastic deformation [1, 2]. Some definition of the rate of plastic deformation is also introduced into approaches which do not use the Mises principle [3].

Suppose the internal energy per unit mass $U = U(E, G, S, \kappa)$ depends on the metric tensor of total deformations G, the elastic-deformation tensor E, the entropy per unit mass S and a certain tensor of the internal variables κ, which will be defined below. The notation used to represent all these quantities is identical, unless otherwise stated, with that used in [1]. References to the formulae derived in [1] are used. Inside the loading surface $\varphi(E, G, S, \kappa) = 0$ the tensors G and E satisfy the equations [1]

$$G^{\cdot} + G \cdot W + W^{\mathrm{T}} \cdot G = 0, \quad W = \partial \mathbf{v}/\partial \mathbf{x}$$
$$E^{\cdot} + E \cdot W - R \cdot E = 0, \quad (\cdot)^{\cdot} = \mathbf{v} \cdot \partial(\cdot)/\partial \mathbf{x} + \partial(\cdot)/\partial t \tag{1}$$

where $\mathbf{v}$ is the rate of deformation vector, and the antisymmetric tensor R is determined by the symmetry conditions of the tensor E (see (8) in [1]).

It is natural to assume that the law of variation of the second-rank symmetric tensor κ in the elastic region is identical with the law of variation of the plastic-deformation tensor P (see (1) and (9) in [1]), i.e. it can be reduced to the same orthogonal transformations

$$\kappa^{\cdot} = R \cdot \kappa - \kappa \cdot R \tag{2}$$

The last equation ensures that the tensor κ is independent of the unloading path and satisfies the requirement of objectivity, i.e. it is invariant under rigid rotations.

Suppose the change to a rotating system of coordinates is described by the orthogonal tensor Q

$$d\mathbf{x}' = Q \cdot d\mathbf{x}, \quad Q \cdot Q^{\mathrm{T}} = \mathrm{I} \tag{3}$$

Using the law of variation of the distortion tensor $A = \partial \mathbf{x}_0/\partial \mathbf{x}$, on changing to a rotating system of coordinates $A = A' \cdot Q$, where $A' = \partial \mathbf{x}_0/\partial \mathbf{x}'$, and Eq. (16) [1]: $A^{\cdot} + A \cdot W = 0$, we obtain $(A')^{\cdot} + A' \cdot W' = 0$, where

$$W' = Q \cdot W \cdot Q^{\mathrm{T}} + Q^{\cdot} \cdot Q^{\mathrm{T}} \tag{4}$$

From (4) and (8) [1] we obtain the law of transformation of the tensor R

$$R' = Q \cdot R \cdot Q^{\mathrm{T}} + Q^{\cdot} \cdot Q^{\mathrm{T}} \tag{5}$$

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Using this equation we obtain

$$(\kappa')^{\cdot} + \kappa' \cdot R' - R' \cdot \kappa' = Q \cdot (\kappa^{\cdot} + \kappa \cdot R - R \cdot \kappa) \cdot Q^{T} \tag{6}$$

where $\kappa' = Q \cdot \kappa \cdot Q^{T}$, which also proves the objectivity of Eq. (2). We can similarly show that Eqs (1) are objective.

The law of conservation of energy for the case of a zero-moment medium and when there are no mass forces can be represented in the form

$$\rho U^{\cdot} = \sigma \cdot\cdot W^{T} - \mathrm{I} \cdot\cdot \partial \mathbf{q}/\partial \mathbf{x} \tag{7}$$

where σ is the Cauchy stress tensor, ρ is the density of the medium, and $\mathbf{q}$ is the heat-flux vector. Equation (7) can be written in terms of the free energy $F(E, G, T, \kappa) = U - TS$ (where T is the absolute temperature; here the function F is not identical with the similar function in [1], since in [1] $F = F(E, P, T)$ in the form

$$TS^{\cdot} + F_{\kappa} \cdot\cdot (\kappa^{\cdot} + \kappa \cdot R - R \cdot \kappa) + \frac{1}{\rho}\left(\mathrm{I} \cdot\cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right) = D; \quad S = -\frac{\partial F}{\partial T} \tag{8}$$

$$D = \frac{1}{\rho}\sigma \cdot\cdot W^{T} - F_G \cdot\cdot G^{\cdot} - F_E \cdot\cdot E^{\cdot} + F_{\kappa} \cdot\cdot (\kappa \cdot R - R \cdot \kappa) \tag{9}$$

Here $F_{\kappa} = \partial F/\partial \kappa$, $F_E = \partial F/\partial E$, $F_G = \partial F/\partial G$ are symmetric second-rank tensors and D has the meaning of the power dissipated per unit mass, which is made up of the power of the internal heat sources $TS^{\cdot} + \rho^{-1}\mathrm{I} \cdot\cdot \partial \mathbf{q}/\partial \mathbf{x}$ and an addition $F_{\kappa} \cdot\cdot (\kappa^{\cdot} + \kappa \cdot R - R \cdot \kappa)$ which has the meaning of the energy expended in unit time in adjusting the internal structure of unit mass of an element. Hence, we have postulated the possibility of introducing a tensor κ such that this energy can be expressed in this way. This can be regarded as a definition of the tensor κ.

Both these components of the power of mechanical-energy dissipation were determined experimentally in [4], where it was noted that existing versions of flow theory give an adequate description of this process for finite deformations.

The condition $D = 0$ for any processes inside the loading surface defines the relation between the tensor σ and the state parameters E, G, T, κ.

The condition for the functional

$$\int_{t_1}^{t_2} D dt \tag{10}$$

to be extremal enables us to obtain equations for E, T and κ in the active region.

Suppose an element of the medium is unloaded. Then, using (1), (2) and (8) of [1], expression (9) for the power dissipated can be converted to the form

$$D = [\rho^{-1}(\sigma - \sigma_1) + 2(G \cdot F_G + E \cdot F_E + \kappa \cdot F_{\kappa})^{a}] \cdot\cdot W^{T} \tag{11}$$

$$\sigma_1 = -\rho\{E \cdot F_E + 2G \cdot F_G + 2(J_1 J_2 - J_3)^{-1}[\Phi^{a} \cdot (J_1^2 E - J_1 E^2) + E \cdot \Phi^{a} \cdot E^2]\}^{c} \tag{12}$$

$$\Phi^{a} = -(2\kappa \cdot F_{\kappa} + E \cdot F_E)^{a} \tag{13}$$

$$J_1 = \mathrm{I} \cdot\cdot E, \quad J_2 = \frac{1}{2}(J_1^2 - E \cdot\cdot E), \quad J_3 = \frac{1}{3}\left(\mathrm{I} \cdot\cdot E^3 + \frac{1}{2}J_1^3 - \frac{3}{2}J_1 E \cdot\cdot E\right) \tag{14}$$

The superscripts c and a denote the symmetric and antisymmetric parts of the tensor, respectively. Since the condition $D = 0$ must be satisfied for any rate of deformation W, we have

$$\sigma = \sigma_1(E, G, T, \kappa), \quad (G \cdot F_G + E \cdot F_E + \kappa \cdot F_{\kappa})^{a} = 0 \tag{15}$$

Here we have taken into account the requirement that the tensor σ must be symmetric. Using the last relation from (13) we have $\Phi = E \cdot F_E + 2G \cdot F_G$, so that the derivatives of the free energy with respect to the components of the tensor κ are eliminated from the expression for σ. The requirement that the stresses must be zero when there are no elastic strains ($E = 1$) and at standard temperature ($T = T_0$) leads to the condition $\Phi^{c}_{E \to \mathrm{I}, T \to T_0} = 0$.

Using the Cayley–Hamilton identity $E^3 - J_1 E^2 + J_2 E - J_3 \mathrm{I} = 0$, expression (12) for the stress tensor can be converted to a form similar to that derived in [1]

$$\sigma = -\rho(J_1 J_2 - J_3)^{-1}[(J_1^2 + J_2)E \cdot \overline{F}_E \cdot E + J_1 J_3 \overline{F}_E + E^2 \cdot \overline{F}_E \cdot E^2 - $$

$$- J_3(E \cdot \overline{F}_E + \overline{F}_E \cdot E) - J_1(E \cdot \overline{F}_E \cdot E^2 + E^2 \cdot \overline{F}_E \cdot E)] \tag{16}$$

$$\overline{F}_E = \partial \overline{F}/\partial E = F_E + E^{-1} \cdot G \cdot F_G + F_G \cdot G \cdot E^{-1} \tag{17}$$

Equation (16) is identical with (13) of [1] if we make the replacement $F \to \bar{F}$, where $\bar{F} = \bar{F}(E, P, T, \kappa) \equiv F(E, E \cdot P \cdot E, T, \kappa)$.

Consider active loading. Assuming that the relationship between the stresses and the strains (15) also holds in this case, we have from the first relation of (1) and (9)

$$D = -F_E \cdot\cdot (E^{\cdot} + E\cdot + EW - R \cdot E) \tag{18}$$

The equations for E, κ and T are obtained from the requirement for the functional (10) to be a maximum in any time interval of the active process under the conditions

$$\tau = TS^{\cdot} + F_\kappa \cdot\cdot(\kappa^{\cdot} + \kappa \cdot R - R \cdot \kappa) + \frac{1}{\rho}\left(\mathrm{I}\cdot\cdot\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right) - D = 0 \tag{19}$$

$$\bar{\varphi}(E, G, T, \kappa) \equiv \varphi(E, G, S(E, G, T, \kappa), \kappa) = 0 \tag{20}$$

$$(E \cdot W - R \cdot E)^a = 0 \tag{21}$$

Equation (21) defines the relation (8) [1] between the tensor R and E and W. Its inclusion in the above conditions enables us to avoid the fairly lengthy differentiation of the tensor $R(E, W)$ with respect to the components of the tensor E.

Hence, it is required to obtain an unconditional extremum of the Lagrange functional

$$\int_{t_1}^{t_2} L(t)dt$$

where

$$L(t) = D - \lambda\bar{\varphi} - \nu\tau - \Lambda \cdot\cdot (E \cdot W - R \cdot E) \tag{22}$$

and Λ is an antisymmetric second-rank tensor. Variation is carried out with respect to the variables E, κ, T and R assuming that G, W and $\partial\mathbf{q}/\partial\mathbf{x}$ are given functions of time.

In this paper we will confine ourselves to the cases of an isothermal process, in which we case we have

$$L = -F_E \cdot\cdot (E^{\cdot} + E \cdot W - R \cdot E) - \lambda\bar{\varphi} - \Lambda \cdot\cdot (E \cdot W - R \cdot E) \tag{23}$$

when the condition $\tau = 0$ defines the influx of heat to the element required to maintain the specified temperature. The tensors E, κ Λ must satisfy Euler's equations

$$\frac{\partial L}{\partial \kappa} = 0, \quad \frac{\partial L}{\partial R} = 0, \quad \left(\frac{\partial L}{\partial E^{\cdot}}\right)^{\cdot} - \frac{\partial L}{\partial E} = 0 \tag{24}$$

When varying the functional the quantities $E(t_1) = E_1$, $E(t_2) = E_2$ are assumed to be given, while the tensor κ is not fixed at the instants of time t_1 and t_2. Since the function (23) is independent of $\kappa^{\cdot}$, the transversatility conditions are satisfied automatically.

Consider the condition $\partial L/\partial\kappa = 0$

$$\frac{\partial^2 F}{\partial\kappa\partial E_{ij}}(E^{\cdot} + E \cdot W - R \cdot E)_{ij} + \lambda\frac{\partial\bar{\varphi}}{\partial\kappa} = 0 \tag{25}$$

This equation defines the elastic strains in the active region only if the fourth-rank tensor $\partial^2F/\partial\kappa\partial E$ is not equal to zero identically, i.e. if, in the expansion of the free energy, there is a dependence on the mixed invariants E and κ. We can assume that in this case

$$\kappa = \kappa(F_E, E, G, T) \tag{26}$$

Using the last relation we can change from the loading function $\bar{\varphi}(E, G, T, \kappa)$ to the function

$$\varphi_1(E, G, T, F_E) \equiv \bar{\varphi}(E, G, T, \kappa(F_E, E, G, T)) \tag{27}$$

Then

$$\frac{\partial\bar{\varphi}}{\partial\kappa} = \left(\frac{\partial\varphi_1}{\partial F_E}\right)_{ij}\frac{\partial(F_E)_{ij}}{\partial\kappa} = \frac{\partial^2 F}{\partial\kappa\partial E_{ij}}\left(\frac{\partial\varphi_1}{\partial F_E}\right)_{ij}$$

and from Eq. (25) we obtain an equation for E for active loading

$$E^{\cdot} + E \cdot W - R \cdot E + \lambda\partial\varphi_1/\partial F_E = 0 \tag{28}$$

Since the third equation of (1) satisfies the objectivity principle, Eq. (28) also satisfies this principle.
The last equation of (24), taking into account relations (27) and (28), gives

$$F_E - [W \cdot (F_E + \Lambda) - (F_E + \Lambda) \cdot R]^c - \lambda \partial \varphi_1 / \partial E = 0 \tag{29}$$

Using (4) and (5) we can show that this equation is objective.
Finally, from the condition $\partial L / \partial R = 0$ we obtain the following equation for determining Λ

$$\Lambda \cdot E + E \cdot \Lambda = F_E \cdot E - E \cdot F_E \tag{30}$$

which is identical in structure with (7) of [1]. Its solution has the form

$$\Lambda = (J_1 J_2 - J_3)^{-1}[(F_E \cdot E - E \cdot F_E)J_1^2 + (E^2 \cdot F_E - F_E \cdot E^2)J_1 - E^2 \cdot F_E \cdot E + E \cdot F_E \cdot E^2] \tag{31}$$

Equations (1) and (28)–(30) together enable us to follow the change in the variables G, E and κ for given rates $W(t)$.

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